**STUDY MATERIAL 1 Module -8 ECONOMICS HONOURS SEMESTER –I CC 1-1 2019-20**

Set Theory

“A set is a Many that allows itself to be thought of as a One.”(Georg Cantor)In the previous chapters, we have often encountered ”sets”, for example,prime numbers form a set, domains in predicate logic form sets as well.Defining a set formally is a pretty delicate matter, for now, we will be happyto consider an intuitive definition, namely:Definition 24.A set is a collection of abstract objects.A set is typically determined by its distinct elements, or members, bywhich we mean that the order does not matter, and if an element is repeatedseveral times, we only care about one instance of the element. We typicallyuse the bracket notation{}to refer to a set.Example 42.The sets{1,2,3}and{3,1,2}are the same, because the or-dering does not matter. The set{1,1,1,2,3,3,3}is also the same set as{1,2,3}, because we are not interested in repetition: either an element is inthe set, or it is not, but we do not count how many times it appears.One may specify a setexplicitly, that is by listing all the elements theset contains, orimplicitly, using a predicate description as seen in predicatelogic, of the form{x, P(x)}. Implicit descriptions tend to be preferred forinfinite sets.Example 43.The setAgiven byA={1,2}is an explicit description. Theset{x, xis a prime number}is implicit.107

Set

A setis a collection of abstract objects–Examples:prime numbers, domain in predicate logic •Determined by (distinct) elements/members. –E.g. {1, 2, 3} = {3, 1, 2} = {1, 3, 2}= {1, 1, 1, 2, 3, 3, 3} •Two common ways to specify a set•Explicit: Enumerate the memberse.g. A= {2, 3}•Implicit: Description using predicates {x|P(x)} e.g. A = {x| x is a prime number}

Membership & Subset

We write x∈Siffx is an element (member) of S. –e.g. A = {x| x is a prime number} then2∈A, 3∈A,5∈A,...,1∉A, 4∉A, 6∉A, ...A set Ais a subsetof the set B, denoted byA ⊆B iffevery element of Ais also an element of B. i.e.,–A⊆B ≜∀x(x∈A→x∈B)–A⊄B ≜¬(A⊆B)≡¬∀x(x∈A→x∈B)≡∃x(x∈A∧x∉B)Subsetversus Membership:S = {rock, paper, scissors}R = {rock},R ⊆S,rock ∈S

109Given a setS, one may be interested in elements belonging toS, or insubset ofS. The two concepts are related, but different.Definition 25.A setAis a subset of a setB, denoted byA⊆B, if andonly if every element ofAis also an element ofB. FormallyA⊆B⇐⇒ ∀x(x∈A→x∈B).Note the two notationsA⊂BandA⊆B: the first one says thatAis a subset ofB, while the second emphasizes thatAis a subset ofB,possibly equal toB. The second notation is typically preferred if one wantsto emphasize that one set is possibly equal to the other.To say thatAis not a subset ofS, we use the negation of∀x(x∈A→x∈B), which is (using the rules we have studied in predicate logic! namelynegation of universal quantifier, conversion theorem, and De Morgan’s law)∃x(x∈A∧x6∈B). The notation isA6⊆B.For an elementxto be an element of a setS, we writex∈S. This is anotation that we used already in predicate logic. Note the difference betweenx∈Sand{x} ⊆S: in the first expression,xis in element ofS, while inthe second, we consider the subset{x}, which is emphasized by the bracketnotation.Example 44.Consider the setS={rock, paper, scissors}, thenR={rock}is a subset ofS, while rock∈S, it is an element ofS.Definition 26.The empty set is a set that contains no element. We denoteit∅or{}.There is a difference between∅and{∅}: the first one is an empty set,the second one is a set, which is not empty since it contains one element: theempty set!Definition 27.The empty set is a set that contains no element. We denoteit∅or{}.Example 45.We say that two setsAandBare equal, denoted byA=B,if and only if∀x,(x∈A↔x∈B).To say that two setsAandBare not equal, we use the negation frompredicate logic, which is:¬(∀x,(x∈A↔x∈B))≡∃x((x∈A∧x6∈B)∨(x∈B∧x6∈A)).

Empty Set

The set that contains no element is called theempty set ornull set.–The empty set is denoted by ∅or by { }.–Note: ∅≠{∅}A=B ≜∀x(x∈A↔x∈B)–Two sets A, B are equal iffthey have the same elements.A≠B ≜¬∀x(x∈A↔x∈B)≡∃x [(x∈A∧x∉B) ∨(x∈B∧x∉A)] –Two sets are not equal if they do not have identical members, i.e., there is some element in one of the sets which is absent in the other.•Example:{1, 2, 3} = {3, 1, 2} = {1, 3, 2}= {1, 1, 1, 2, 3, 3, 3}

Set Equality

111This makes our earlier example{1,2,3}={1,1,1,2,3,3,3}easier tojustify than what we had intuitively before: both sets are equal becausewhenever a number belongs to one, it belongs to the other.Definition 28.The cardinality of a setSis the number of distinct elementsofS. If|S|is finite, the set is said to be finite. It is said to be infiniteotherwise.We could say the number of elements ofS, but then this may be confusingwhen elements are repeated as in{1,2,3}={1,1,1,2,3,3,3}, while thereis no ambiguity for distinct elements. There|S|=|{1,2,3}|= 3. The setof prime numbers is infinite, while the set of even prime numbers is finite,because it contains only 2.Definition 29.The power setP(S) of a setSis the set of all subsets ofS:P(S) ={A, A⊆S}.IfS={1,2,3}, thenP(S) containsSand the empty set∅, and allsubsets of size 1, namely{1},{2}, and{3}, and all subsets of size 2, namely{1,2},{1,3},{2,3}.The cardinality ofP(S) is 2nwhen|S|=n. This is not such an obviousresult, it may be derived in several ways, one of them being the so-calledbinomial theorem, which says that(x+y)n=n∑j=0(nj)xjyn−j,where(nj)counts the number of ways to choosejelements out ofn. Thenotation∑nj=0means that we sum for the values ofjgoing from 0 ton. SeeExercise 33 for a proof of the binomial theorem. Whenn= 3, evaluating inx=y= 1, we have23=(30)+(31)+(32)+(33)and we see that(30)says we pick no element from 3, there is one way, and itcorresponds to the empty set, then(31)is telling us that we have 3 ways tochoose a single subset, this is for{1},{2}, and{3},(32)counts{1,2},{1,3},{2,3}and(33)counts the whole set{1,2,3}.When dealing with sets, it is often useful to draw Venn diagrams toshow how sets are interacting. They are useful to visualize “unions” and“intersections”.

Cardinality

The cardinality|S| of Sisthe number of elements in S. –e.g. for S={1, 3}, |S| =2If |S| is finite, S is a finite set; otherwise, S is infinite. –The set of positiveintegers is an infinite set.–The set of primenumbers is an infinite set.–The set of even primenumbers is a finite set.•Note: |∅| = 0

Power Set

The power set P(S) of a given set S is the set of all subsets of S: P(S) = { A | A ⊆S}.•Example–For S= {1,2,3} P(S)={∅,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}•If a set A has n elements, then P(s) has 2n elements. •Hint: Try to leverage the Binomial theorem

Venn Diagram

A Venn diagram is used to show/visualize the possible relations among a collection of sets.John Venn(1834-1923)Pictures from Union and Intersection

The union of sets A and B is the set of those elements that are either in A or in B, or in both.A∪B≜{x | x∈A∨x∈B}BAThe intersection of the sets A and Bis the set of all elements that are in both A and B.A∩B≜{x | x∈A∧x∈B}BAA∩B

Disjoint Sets

Sets Aand Bare disjointiffA∩B= ∅–|A ∩B| = 0Lions ∩Fishes = ∅LionsFishesWhat about me?© photographer

Cardinality Of Union⏐A∪B⏐= ⏐A ⏐+ ⏐B ⏐-⏐A∩B⏐

115Definition 30.The union of the setsAandBis by definitionA∪B={x, x∈A∨x∈B}.The intersection of the setsAandBis by definitionA∩B={x, x∈A∧x∈B}.When the intersection ofAandBis empty, we say thatAandBaredisjoint.The cardinality of the union and intersection of the setsAandBarerelated by:|A∪B|=|A|+|B|−|A∩B|.This is true, because to count the number of elements inA∪B, we startby counting those inA, and then add those inB. IfAandBwere disjoint,then we are done, otherwise, we have double counted those in both sets, sowe must subtract those inA∩B.Definition 31.The difference ofAandB, also called complement ofBwithrespect toAis the set containing elements that are inBbut not inB:A−B={x, x∈A∧x6∈B}.The complement ofAis the complement ofAwith respect to the universeU: ̄A=U−A={x, x6∈A}.The universeUis the set that serves as a framework for all our set compu-tations, the biggest set in which all the other sets we are interested in lie.Note that ̄A=A.Definition 32.The Cartesian productA×Bof the setsAandBis the setof all ordered pairs (a,b), wherea∈A,b∈B:A×B={(a,b), a∈A∧b∈B}.Example 46.TakeA={1,2},B={x,y,z}. ThenA×B={(a,b), a∈{1,2}∧b∈{x,y,z}}thusacan be either 1 or 2, and for each of these 2 values,bcan be eitherx,yorz:A×B={(1,x),(1,y),(1,z),(2,x),(2,y),(2,z)}.Note thatA×B6=B×A, and that a Cartesian product can be formedfromnsetsA1,...,An, which is denoted byA1×A2×···×An.

Set Difference & Complement

The difference of A and B(or complement of B with respect to A)is the set containing those elements that are in A but not in B.A −B ≜{x | x∈A∧x∉B}Thecomplement of A is the complement of A with respect to U.≜{x | x∉A}A-UA=AU

Cartesian Product The Cartesian product AxBof the sets A and B is the set of all ordered pairs (a,b) where a ∈A and b ∈B. A ×B ≜{(a,b) | a ∈A ∧b ∈B}•Example: A = {1,2}, B = {x,y,z}A ×B = {(1,x), (1,y), (1,z), (2,x), (2,y), (2,z)}B ×A = {(x,1), (x,2), (y,1), (y,2), (z,1), (z,2)}•In general: A1×A2 ×...×An≜{(a1,a2, ... , an) | ai∈Aifor i=1,2, ..., n}•| A1×A2 ×...×An| = |A1| |A2 | ...|An|